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Change in Dispersion Function from Field Gradient Errors

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Method

We consider changes in the momentum dispersion function induced by field gradient errors of quadrupole magnets located around a ring.

The gradient error in the quadrupole magnet located at $s=s_i$ is represented by a single thin-lens and its strength is specified as

$$k_i = \frac{\Delta B' l}{B\beta} \quad (1)$$

where $B\beta$ is the magnetic rigidity. The momentum dispersion function $\gamma(s)$ of such a machine should satisfy the equation

$$\ddot{\gamma}(s) + [K(s) + \sum_{i=1}^N k_i \delta(s-s_i)] \dot{\gamma}(s) = \Omega(s), \quad (2)$$

where $K(s)$ and $\Omega(s)$ are periodic focussing and bending functions, respectively, and N denotes a total number of quadrupole magnets with a field gradient error; in general

$|K(s_i)| \gg |k_i|$. We express the periodic solution of (2) in the form

$$\gamma(s) = \gamma^{(0)}(s) + \gamma^{(1)}(s) + \gamma^{(2)}(s) + \dots, \quad (3)$$

where superscripts stand for the order of expansion. Substituting (3) into (2) and equating terms of the same order, we obtain

$$0(0) \quad \ddot{\gamma}^{(0)} + K(s) \dot{\gamma}^{(0)} = \Omega(s), \quad (4a)$$

$$0(1) \quad \ddot{\gamma}^{(1)} + K(s) \dot{\gamma}^{(1)} = \sum_{i=1}^N (-k_i) \delta(s-s_i) \gamma^{(0)}(s), \quad (4b)$$

$$0(2) \quad \ddot{\gamma}^{(2)} + K(s) \dot{\gamma}^{(2)} = \sum_{i=1}^N (-k_i) \delta(s-s_i) \gamma^{(1)}(s), \quad (4c)$$

$$\vdots \quad \vdots \quad \vdots \\ 0(n+1) \quad \ddot{\gamma}^{(n+1)} + K(s) \dot{\gamma}^{(n+1)} = \sum_{i=1}^N (-k_i) \delta(s-s_i) \gamma^{(n)}(s), \quad (4d)$$

The periodic solution of (4d) is

$$\begin{aligned} \gamma^{(n+1)}(s) &= \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \int_s^{s+C} \sqrt{\beta(s')} \sum_{i=1}^N (-k_i) \delta(s-s_i) \gamma^{(n)}(s') \cos[\nu(\phi(s)-\phi(s')+\pi)] ds' \\ &= \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \sum_{i=1}^N (-k_i) \sqrt{\beta(s_i)} \gamma^{(n)}(s_i) \cos[\nu(\phi(s)-\phi(s_i)+\pi)], \end{aligned} \quad (5)$$

where $\phi(s)$ is defined by

$$\phi(s) = \int_s^s \frac{ds'}{\sqrt{\beta(s')}} ,$$

$\beta(s)$ and ν are a betatron function and tune, respectively.

Introducing a $(1 \times N)$ matrix

$$\Delta(s) = \|\delta_p\| ,$$

with

$$\delta_p = \frac{\sqrt{\beta(s) \beta(s_p)}}{2 \sin \pi v} (-k_p) \cos [\nu(\phi(s) - \phi(s_p) + \pi)], \quad (6)$$

we write (5) in the matrix form

$$\gamma^{(n+1)}(s) = \Delta(s) \cdot X^{(n)}, \quad (7)$$

with

$$X^{(n)} = (\gamma^{(n)}(s_1), \gamma^{(n)}(s_2), \dots, \gamma^{(n)}(s_N))^T.$$

Linearity of (5) with respect to $\gamma(s_i)$ leads to a recursion relation for $\gamma^{(n)}(s_k)$ in the matrix form

$$X^{(n)} = M \cdot X^{(n-1)}, \quad (8)$$

where

$$M = \| w_{p,q} \|,$$

with

$$w_{p,q} = \frac{\sqrt{\beta(s_p) \beta(s_q)}}{2 \sin \pi v} (-k_q) \cos [\nu(\phi(s_q) - \phi(s_p) + \pi)]. \quad (9)$$

From (8), we get

$$X^{(n)} = M^n \cdot X^{(0)}. \quad (10)$$

Substituting this expression for $X^{(n)}$ into (7), we obtain

$$\gamma^{(n+1)}(s) = \Delta(s) \cdot M^n \cdot X^{(0)} \quad (11)$$

and, from (3),

$$\gamma(s) = \gamma^{(0)}(s) + \Delta(s) \cdot [M^0 + M^1 + M^2 + \dots] \cdot X^{(0)} \quad (12)$$

If this matrix power series converges, Eq.(12) is equivalent to a simple form

$$\gamma(s) = \gamma^{(0)}(s) + \Delta(s) \cdot (I - M)^{-1} \cdot X^{(0)}, \quad (13)$$

where I is the unit matrix of order N.

Examples

N=1: From (6) and (9), we have

$$\delta_1 = \frac{1}{2} \sqrt{\beta(s)\beta(s_1)} \cdot \frac{\cos [\nu(\phi(s)-\phi(s_1)+\pi)]}{\sin \pi\nu} (-k_1), \quad (a1)$$

$$m_{11} = \frac{1}{2} \beta(s_1) \cdot \frac{\cos \pi\nu}{\sin \pi\nu} (-k_1), \quad (a2)$$

then,

$$\gamma(s) = \gamma^{(0)}(s) + \frac{\delta_1}{1-m_{11}} \gamma^{(0)}(s_1). \quad (a3)$$

N=2:

matrix elements

$$\delta_p = \frac{1}{2} \sqrt{\beta(s)\beta(s_p)} \cdot \frac{\cos [\nu(\phi(s)-\phi(s_p)+\pi)]}{\sin \pi\nu} (-k_p) \quad (b1)$$

(p=1, 2)

$$\left\{ \begin{array}{l} m_{p,p} = \frac{1}{2} \beta(s_p) \cdot \cot \pi\nu \cdot (-k_p), \\ m_{p,q} = \frac{1}{2} \sqrt{\beta(s_p)\beta(s_q)} \cdot \frac{\cos [\nu(\phi(s_q)-\phi(s_p)+\pi)]}{\sin \pi\nu} (-k_q). \end{array} \right. \quad (b2)$$

Then, the matrices of Δ and $(I-M)^{-1}$ become

$$\Delta(s) = \frac{(-1)\sqrt{\beta(s)}}{2\sin\pi v} (k_1\sqrt{\beta_1}\cos(\pi v + \psi_{s1}), k_2\sqrt{\beta_2}\cos(\pi v + \psi_{s2})), \quad (b3)$$

$$(I-M)^{-1} = \frac{1}{D} \begin{pmatrix} 1 + \frac{k_2}{2}\beta_2 \cot\pi v & -\frac{k_2}{2}\sqrt{\beta_1\beta_2} \frac{\cos(\pi v + \psi_{s1})}{\sin\pi v} \\ -\frac{k_1}{2}\sqrt{\beta_1\beta_2} \frac{\cos(\pi v - \psi_{s1})}{\sin\pi v} & 1 + \frac{k_1}{2}\beta_1 \cot\pi v \end{pmatrix}, \quad (b4)$$

with

$$\beta_i = \beta(s_i),$$

$$\psi_{ij} = v(\Phi(s_i) - \Phi(s_j)),$$

$$D = 1 + \frac{\cot\pi v}{2} (k_1\beta_1 + k_2\beta_2) + \frac{\beta_1\beta_2}{4} \frac{k_1k_2}{\beta_1\beta_2} \frac{\sin^2\psi_{s1}}{\sin^2\pi v}. \quad (b5)$$

From (b3), (b4) and (13),

$$\begin{aligned} \gamma(s) = \gamma^{(0)}(s) &+ \frac{(-1)\sqrt{\beta(s)}}{2D\sin\pi v} \left\{ \gamma^{(0)}(s_1) \left[k_1\sqrt{\beta_1}\cos(\pi v + \psi_{s1}) + \frac{k_1k_2\sqrt{\beta_1}\beta_2}{4\sin\pi v} P_1(s) \right] \right. \\ &\left. + \gamma^{(0)}(s_2) \left[k_2\sqrt{\beta_2}\cos(\pi v + \psi_{s2}) + \frac{k_1k_2\beta_1\sqrt{\beta_2}}{4\sin\pi v} P_2(s) \right] \right\}, \end{aligned} \quad (b6)$$

with

$$P_1(s) = \cos(2\pi v + \psi_{s1}) - \cos(2\pi v - \psi_{s1} + \psi_{s2}),$$

$$P_2(s) = \cos(2\pi v + \psi_{s2}) - \cos(2\pi v + \psi_{s1} + \psi_{s2}),$$

$$D = 1 + \frac{\cot\pi v}{2} (k_1\beta_1 + k_2\beta_2) + \frac{k_1k_2}{4} \frac{\beta_1\beta_2}{\beta_1\beta_2} \frac{\sin^2\psi_{s1}}{\sin^2\pi v}.$$

Discussions

When field gradient errors are caused by power supply ripples, for example, we have

$$k_i = \frac{B'_i l_i}{B_S} \times 10^{-\alpha} \quad (14)$$

where B'_i and l_i are the field gradient and length of the i -th quadrupole magnet, respectively, and α represents the magnitude of the power supply ripple. All quadrupoles are assumed to be in series with the same power supply.

If $|m_{p,q}| \ll 1$, contributions of higher order terms may be negligible. For such a case, we can write the solution of perturbed equation (2) to the first order, that is,

$$\gamma(s) = \gamma^{(0)}(s) + \sum_{i=1}^N \delta_i \gamma^{(0)}(s_i). \quad (15)$$

Let apply the considerations to the \bar{P} Accumulator Ring¹. Parameters relevant to the present discussion are listed in Table 1. The parameter $m_{p,q}$ satisfies the inequality

$$\begin{aligned} |m_{p,q}| &< \frac{(\beta_i)_{\max} (B'_i)_{\max} (l_i)_{\max}}{2 \sin \pi v \cdot B_S} \times 10^{-\alpha} \\ &= \frac{30 \cdot 103 \cdot 1.3}{2 \sin (\pi \cdot 6.61) \cdot 294} \times 10^{-\alpha} \\ &= 7.2756 \times 10^{-\alpha}. \end{aligned} \quad (16)$$

Assuming $\alpha \geq 2$, we see that the first-order solution (15) is accurate enough for our purpose. We can calculate the change in the dispersion function at any point in the ring from a convenient form

$$\gamma(s) = \gamma^{(0)}(s) + \frac{(-1)^{\sqrt{\beta(s)}}}{2 \sin \pi v B_S} 10^{-\alpha} \sum_{i=1}^N \sqrt{\beta_i} B'_i l_i \cos[\pi(v - 2\Delta v_i)] \gamma_i^{(0)} \quad (17)$$

An amount of change calculated at the particular position ($s=0$) where the dispersion is required to be zero with high precision is compared with the Synch output²;
for $\alpha = 5$

Calculation: $\gamma(0) = \gamma^{(0)}(0) + (-1.89 \times 10^{-4} m),$

Synch: $\gamma(0) = \gamma^{(0)}(0) + (-2. \times 10^{-4} m).$

Here only gradient errors in large aperture quadrupole magnets are assumed.

References

1. " Design Report Tevatron I Project ", Fermilab, Oct. 1982.
2. D.Johnson, private communications, April(1983).

TABLE 1

NO. OF QMAG	BX(M)	B(KG/M)	L(M)	DNUE*	EX*(M)
1	16.2936	103.8087	.6401	.1410	.0918
2	12.1482	-103.8087	1.3117	.1590	.0680
3	27.6176	103.8087	.7010	.1770	.0841
4	15.7332	96.6333	.4572	.2490	.6465
5	2.8710	-97.4126	.8280	.3500	.5935
6	27.8812	96.6333	.7010	.5330	1.7353
7	6.3068	-97.4126	.7010	.5930	.5340
8	21.0945	96.6333	.4572	.7280	-.1799
9	7.7138	-97.4126	.4572	.8170	1.1236
10(QL)	24.7911	40.8765	.4572	.8970	5.3016
11(QL)	30.0168	89.3989	.8738	.9350	8.9183
12(QL)	15.9959	-89.3989	.7722	.9450	7.1831
13(QL)	12.2270	-89.3989	.7722	.9630	7.3121
14(QL)	15.8280	89.3989	.6426	.9760	9.1719
15(QL)	15.8280	89.3989	.6426	1.2280	9.1719
16(QL)	12.2270	-89.3989	.7722	1.2410	7.3121
17(QL)	15.9959	-89.3989	.7722	1.2590	7.1831
18(QL)	30.0168	89.3989	.8738	1.2690	8.9183
19(QL)	24.7911	40.8765	.4572	1.3070	5.3016
20	7.7138	-97.4126	.4572	1.3870	1.1236
21	21.0945	96.6333	.4572	1.4760	-.1799
22	6.3068	-97.4126	.7010	1.6110	.5340
23	27.8812	96.6333	.7010	1.6710	1.7353
24	2.8710	-97.4126	.8280	1.8540	.5935
25	15.7332	96.6333	.4572	1.9550	.6465
26	27.6176	103.8087	.7010	2.0270	.0841
27	12.1482	-103.8087	1.3117	2.0450	.0680
28	16.2936	103.8087	.6401	2.0630	.0918
29	16.2936	103.8087	.6401	2.3450	.0918
30	12.1482	-103.8087	1.3117	2.3630	.0680
31	27.6176	103.8087	.7010	2.3810	.0841
32	15.7332	96.6333	.4572	2.4530	.6465
33	2.8710	-97.4126	.8280	2.5540	.5935
34	27.8812	96.6333	.7010	2.7370	1.7353
35	6.3068	-97.4126	.7010	2.7970	.5340
36	21.0945	96.6333	.4572	2.9320	-.1799
37	7.7138	-97.4126	.4572	3.0210	1.1236
38(QL)	24.7911	40.8765	.4572	3.1010	5.3016
39(QL)	30.0168	89.3989	.8738	3.1390	8.9183
40(QL)	15.9959	-89.3989	.7722	3.1490	7.1831
41(QL)	12.2270	-89.3989	.7722	3.1670	7.3121
42(QL)	15.8280	89.3989	.6426	3.1800	9.1719
43(QL)	15.8280	89.3989	.6426	3.4320	9.1719
44(QL)	12.2270	-89.3989	.7722	3.4450	7.3121
45(QL)	15.9959	-89.3989	.7722	3.4630	7.1831
46(QL)	30.0168	89.3989	.8738	3.4730	8.9183
47(QL)	24.7911	40.8765	.4572	3.5110	5.3016

NO.	OF QMAG	BX(M)	B(KG/M)	L(M)	DNUE*	EX(M)
48		7.7138	-97.4126	.4572	3.5910	1.1236
49		21.0945	96.6333	.4572	3.6800	-.1799
50		6.3068	-97.4126	.7010	3.8150	.5340
51		27.8812	96.6333	.7010	3.8750	1.7353
52		2.8710	-97.4126	.8280	4.0580	.5935
53		15.7332	96.6333	.4572	4.1590	.6465
54		27.6176	103.8087	.7010	4.2310	.0841
55		12.1482	-103.8087	1.3117	4.2490	.0680
56		16.2936	103.8087	.6401	4.2670	.0918
57		16.2936	103.8087	.6401	4.5490	.0918
58		12.1482	-103.8087	1.3117	4.5670	.0680
59		27.6176	103.8087	.7010	4.5850	.0841
60		15.7332	96.6333	.4572	4.6570	.6465
61		2.8710	-97.4126	.8280	4.7580	.5935
62		27.8812	96.6333	.7010	4.9410	1.7353
63		6.3068	-97.4126	.7010	5.0010	.5340
64		21.0945	96.6333	.4572	5.1360	-.1799
65		7.7138	-97.4126	.4572	5.2250	1.1236
66(QL)		24.7911	40.8765	.4572	5.3050	5.3016
67(QL)		30.0168	89.3989	.8738	5.3430	8.9183
68(QL)		15.9959	-89.3989	.7722	5.3530	7.1831
69(QL)		12.2270	-89.3989	.7722	5.3710	7.3121
70(QL)		15.8280	89.3989	.6426	5.3840	9.1719
71(QL)		15.8280	89.3989	.6426	5.6360	9.1719
72(QL)		12.2270	-89.3989	.7722	5.6490	7.3121
73(QL)		15.9959	-89.3989	.7722	5.6670	7.1831
74(QL)		30.0168	89.3989	.8738	5.6770	8.9183
75(QL)		24.7911	40.8765	.4572	5.7150	5.3016
76		7.7138	-97.4126	.4572	5.7950	1.1236
77		21.0945	96.6333	.4572	5.8840	-.1799
78		6.3068	-97.4126	.7010	6.0190	.5340
79		27.8812	96.6333	.7010	6.0790	1.7353
80		2.8710	-97.4126	.8280	6.2620	.5935
81		15.7332	96.6333	.4572	6.3630	.6465
82		27.6176	103.8087	.7010	6.4350	.0841
83		12.1482	-103.8087	1.3117	6.4530	.0680
84		16.2936	103.8087	.6401	6.4710	.0918

$$B_P = 294.0756 \text{ (KG-m)}$$

NUEX = 6.61177

* : VALUES AT THE CENTER OF EACH MAGNET

QL : LARGE APERTURE QUADRUPOLE MAGNET